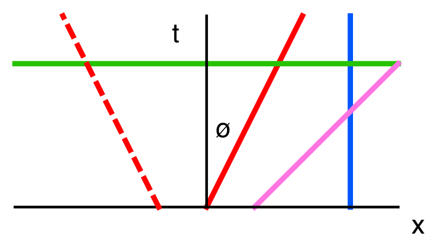
Spacetime Diagrams I

Just like we used graphs to help visualize the relationships between position, velocity and acceleration, we can use graphs to visualize concepts in relativity. Since relativity is all about how different inertial frames measure the world, we will make diagrams that allow us to visualize two inertial frames at the same time. We call these "Spacetime Diagrams."

It was one of Einstein's math professors, Minkowski, who in 1908 first suggested reframing the theory of relativity in terms of a four-dimensional "spacetime" and how one can graphically interpret the ideas of relativity. Einstein initially balked at Minkowski's work, but it turned out to be the key that let Einstein finish his theory of gravity that we call general relativity.

There are lots of ways to make a spacetime diagram, depending on what units one wants to use and also what one is trying to do with the diagram. For this handout, we will stick with a pretty standard form.

Basically, a spacetime diagram is just a time vs position graph with some picky units. We will do this handout using only one space dimension. It is also easy to make them with two space dimensions (and then time would be the third vertical dimension.) Our very first spacetime diagram is shown below:



The blue vertical line would represent something whose position doesn't change i.e, an object at rest. This would be the "worldline" of an object that is not moving in this reference frame.

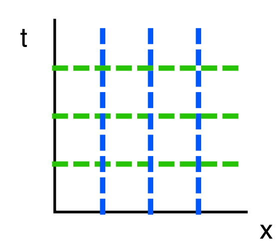
The solid red line would represent something that was moving to the right with a constant speed – and its speed would be related to the angle ø shown. This would be the "worldline" of an object that is moving in this reference frame. Since the speed of that object is "distance / time" the speed of the object is just

$$v = \tan ø$$

The dashed red line would be an object that is moving to the left with a constant speed.

The green horizontal line would represent a certain time in this reference frame and would not be a physical thing.

The pink line moving up at 45° is special with the right units. To really make a spacetime diagram useful, we need to build light into it from the start, and that is by the choice of units to measure space and time. We could just use geometric units, so that both time and space are measured in meters. We could also use light-seconds and seconds, or some equivalent set. Notice that in either case, the speed of light is 1, and so the slope of a "time vs postion" graph would be 1, and so a ray of light would have a slope of 1. This is the key to a useful spacetime diagram. The pink line moving up with a slope of 1 and an angle of 45° represents a ray of light.



Look at the diagram to the right. The green horizontal lines would each be a line showing a particular time in this reference frame. The blue vertical lines would show particular positions. It's just cartesian coordinates like you know and love. I just want to point out that you should think of the lines of constant time as being

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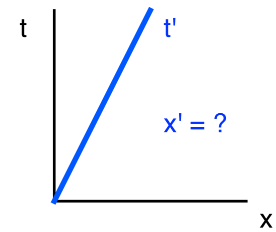
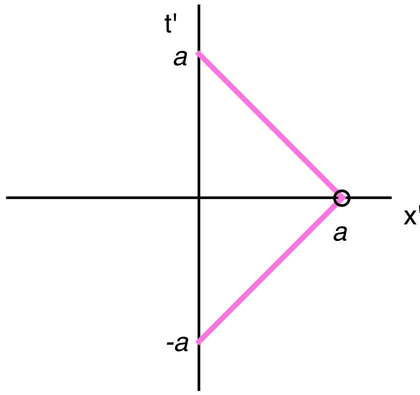
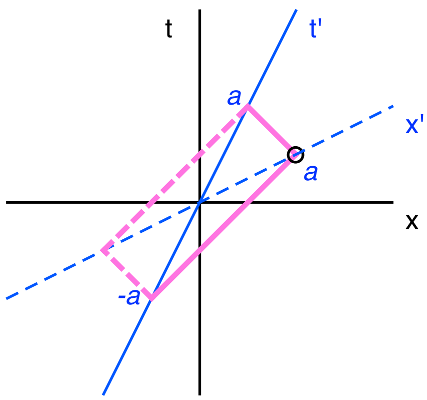
parallel to the x axes (which itself is really everywhere where time is zero.) Likewise, the blue lines of constant position are all parallel to the time axes (which itself is just all the times when the position is zero.) You'll see why I am saying this in a little bit.

The real value of the spacetime diagram is to visualize two different reference frames and how they both measure the world. The black t and x axes are the initial reference frame, which we will call S. In the diagram to the right, the blue line shows something moving to the right in S. Let's consider that to be the origin of our second reference frame (if you want to call it an observer moving to the right, that would be correct.) Since that is the location of the origin of the second reference frame, that is the time axis of S', which we will label as t'. Now we need to figure out what happens to the x' axis and how to show it in our diagram.

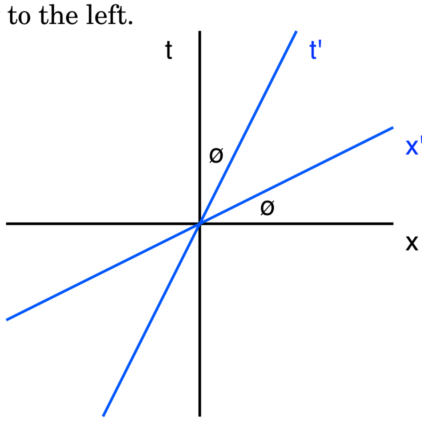


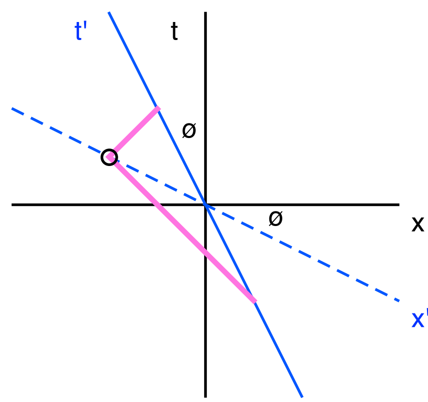
The key is to define a way of finding the x-axis given a t-axis and using light. So let's first jump into the S' frame for a moment do just that. We will define the x' axis the following way. Imagine a photon is emitted to the right at some time -*a*. It spreads out (moving up at 45° in our spacetime diagram) and at time 0 is at location *a*. (Remember that the speed of light is 1.) At that point, imagine a second photon is emitted and heads back to the left. The second photon will cross the time axis at a time of *a*. It was just a round trip of light between the origin and some position along the x-axis. We could map out the entire position axis by just changing the value of *a*. Let's do that back in the S frame.





The diagram to the left shows the situation as seen in S. A photon is emitted from t' = -a, and a second photon is received at t' = a. Since light moves at a 45° angle, we just find where the emitted light and received light paths cross. That must be the position x' = a. So the x' axis is shown as the blue dotted line. Notice the symmetry of the S' axes. The pink dotted lines would have shown light initially sent to the left and then return, and results in a pick rectangle, in which the S' axes are diagonals. Therefor the angle between the two time axes is the same as the angle between the two position axes. This is shown in the diagram to the bottom left. A similar construction is shown to the bottom right which will show a reference frame that is moving to the left.





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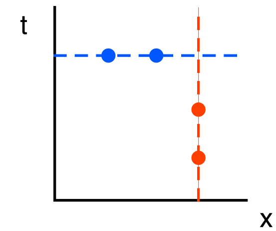
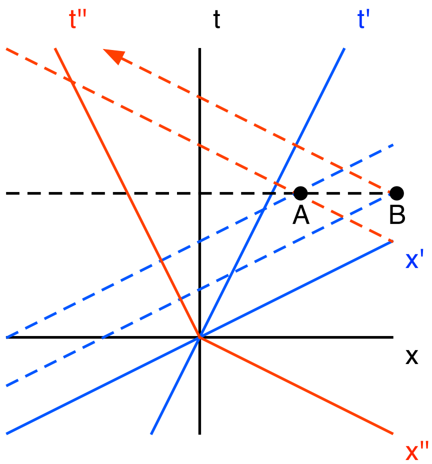
Let's start using our spacetime diagram. In the diagram to the right, there are four events plotted. All four events have their own coordinates in spacetime. An event could be a collision, a light turning on, someone saying "hi," – really anything, which is why we use such a generic term. But all events happen at a particular point in space and time – hence at a particular point in spacetime. The two blue events in the diagram happen at the same time in this frame, but at different locations, while the two red events happen at the same location but at different times.



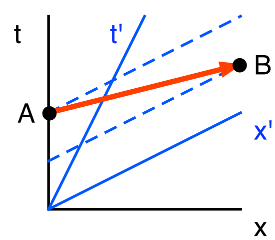


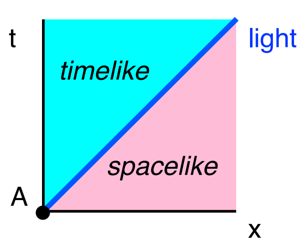
The diagram to the left shows how 2 independent events that are simultaneous in one frame are not simultaneous in another. Events A and B are simultaneous in the S frame. The S' frame is moving to the right relative to S and is shown in blue while the S" frame is moving to the left and is shown in red. The dashed lines are the time coordinates in each frame. Notice how A happens before B in S", while B happens before A in S'.

There is only one reference in which two events separated in space can be simultaneous. In all other reference frames, the order of events depends on the relative motion, and it turns out that the difference in times of the two events depends on the relative speeds.

It needs to be stressed that the two events in the above diagram cannot depend on each other, meaning event A cannot cause event B (or vice versa.) If in the above diagram, A was a button that triggers an explosion at B, look at the consequences for each of the reference frames. In S", A would happen first, then B; the trigger causes the explosion like it should. Tbe S frame could also be somewhat ok, but the trigger and the explosion are simultaneous. In the S' frame however, B happens before A, so that means in that reference frame the explosion happens before the trigger; the effect happens after the cause. Since the cause should happen before the effect, there can be no instantaneous communication between two distant events.

The causality problems just described are not limited to instantaneous communication. The exact same issue arises if something (signal or object) could travel faster than light. For example, in the diagram to the right, imagine an object travels faster than light from A to B. In the S frame, it looks fine; it leaves A before it arrives at B. In the S' frame though, it is backwards, and it arrives before it leaves. The *causality* between events is broken if anything can travel faster than light.





Two events that are a causal, meaning one event could cause the next event, are said to be *timelike*. Timelike events are separated mostly by time. In the diagram to the left, if event A happens at the origin, any other event that happens above the light world line (shaded light blue) could be caused by A. Notice that an observer could physically be at both events. Below the light worldline, shaded pink, would be *spacelike*. Any second event that is spacelike could *not* be caused by the event at A. Two events that are spacelike are separated mostly by space, and it is impossible for an observer to be physically present at both events.

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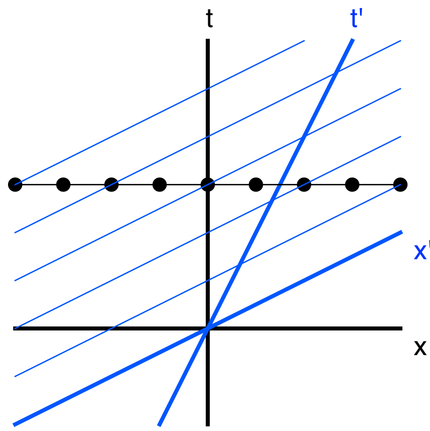
Finally, let's generalize the issue of simultaneity to how relative motion affects time. Time is wrapped up in space. What is uniform time in one reference frame is not uniform time in another. To see this, imagine we just set up a string of clocks equally spaced out in our reference frame. Each dot in the spacetime diagram to the right would be one of those clocks. In our reference frame (S), they are all synchronized – they are all on the same time axis in the S frame. That means for example, they all hit 1:00 simultaneously in our reference frame. (Please remember that we would have to *calculate* that was the case: we would literally *see* the clocks closest to use first hit 1:00, the farther away a clock was, the longer it would take light to travel from the clock to us, but knowing the distances and the speed of light we would conclude they all hit 1:00 simultaneously.) What would someone traveling by us to the right conclude? The S' is shown in blue, with several lines of constant time for the S' frame drawn in. In the S' frame, the clocks do not hit 1:00 simultaneously. The right most clock would hit 1:00 first, and then they would go in order from right to left, with a constant separation that would depend on the speed of the S' frame.



What would happen to clocks synchronized in the S' frame? That is shown in the diagram to the right. This time, the blue dots are all on the same time axis in the S' frame, so in the S' frame they are all synchronized. Several time axes for the S frame are also shown. Notice how in the S frame the clock on the left happens first, then they go in order from left to right. Synchronized clocks going past us to the right are not synchronized to us – the clocks to the left are ahead of the clocks to the right.